TECHNICAL RESEARCH STATEMENT (2024)

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1. INTRODUCTION

My research program explores relationships between the combinatorics and geometry of flag varieties, Schubert varieties and Coxeter groups. For over a century, Schubert varieties have been studied due to their rich combinatorial and geometric structures. Schubert calculus in relation to enumerative geometry is the main focus of Hilbert's 15th problem. Today, the study of Schubert geometry and combinatorics remains a very active field of mathematics and there are many open questions regarding Schubert varieties.

My research falls into three categories:

- **The geometry and combinatorics of Schubert varieties:** This research program involves studying the combinatorial and geometric aspects of Schubert varieties. I have worked on many projects which include the isomorphism problem on Schubert varieties, studying fiber bundle structures, calculating the Nash blow-up, and enumerating smooth/rationally smooth Schubert varieties. Other projects include exploring the connections between Schubert varieties and permutation pattern avoidance. More details on this research are given in Section 2.

- **Schubert Calculus and its applications:** This research program involves a variety of different aspects of Schubert calculus. My work on this topic include studying Schubert calculus for Kac-Moody flag varieties, exploring saturation properties for T-equivariant cohomology, looking at recursive structures of the Belkale-Kumar product and finding applications of Schubert calculus to problems in frame theory and functional analysis. I have also studied non-commutative Littlewood-Richardson coefficients which correspond to a non-commutative analogue of Schubert calculus in the Grassmannian. More details on this research are given in Section 3.

- **Other research:** I have also worked on several related research projects that fall outside the two categories above. These include projects on the combinatorial properties of Springer fibers, studying generating functions on intervals in Young's lattice, and developing algorithms to efficiently calculate the Demazure product on permutations. More details on this work is given in Section 4.

2. GEOMETRY OF SCHUBERT VARIETIES

Let G be a Lie group over an algebraically closed field and let W denote the Weyl group of G. A Schubert variety is the closure of a B-orbit (Borel subgroup-orbit) in the flag manifold G/B. The Weyl group indexes these orbits, so for any $w \in W$, define the Schubert variety

$$X(w) := \overline{BwB}/B.$$

The combinatorial properties of W are closely related to the geometry of Schubert varieties. For example, the Poincaré series of $X(w)$ is the rank generating function of the Bruhat interval $[e, w]$. In the following sections, I describe several research projects on the geometry and combinatorics of Schubert varieties.

2.1. **The isomorphism problem.** In [41], Slofstra and I consider the following classification problem that is central to the field of algebraic geometry.

Question 2.1. *When are two Schubert varieties are algebraically isomorphic?*

We answer this question by defining the combinatorial notion of *Cartan equivalence*. Given $w, w' \in W$, a Cartan equivalence $w \sim w'$ is a bijection between the support sets of w and w' that simultaneously matches the reduced word structure and Cartan data of w and w'. We prove the following:

Theorem 2.2. [41, Theorem 1.3] *The Schubert varieties $X(w)$ and $X(w')$ are algebraically isomorphic if and only if there is a Cartan equivalence $w \sim w'$.*

One surprising consequence Theorem 2.2 is that isomorphism classes of Schubert varieties in a given flag variety are governed by graph automorphisms of the underlying Dynkin diagram. For example, if $X(w)$ is fully supported of type A_n, then it is isomorphic to at most one other Schubert variety of the same type. This is due to the fact that the type A Dynkin diagram is a path which has only one non-trivial graph automorphism. Theorem 2.2 is versatile in the sense that it applies to the general class of Kac-Moody Schubert varieties. It can also be used to compare Schubert varieties of across different types. In [36], Tarigradschi, Xu and I give an analogous isomorphism criterion for cominuscule Schubert varieties in terms of labelled posets.

2.2. **Fiber bundle structures.** Let P be parabolic subgroup of G. The projection map

$$\pi : G/B \to G/P$$

gives a P/B-fiber bundle structure on the flag variety G/B. If W_P denotes the Weyl group P, then there is a unique parabolic decomposition of an element $w = vu$ where $u \in W_P$ and v is minimal length in the coset wW_P. Restricting the projection π to the Schubert variety $X(w)$ yields the projection

$$\pi : X(w) \to X^P(v)$$

where the generic fiber is isomorphic to $X(u)$. However π does not usually induce a fiber bundle structure on $X(w)$.

Question 2.3. *When is π restricted to $X(w)$ an $X(u)$-fiber bundle?*

In [38], Slofstra and I answer this question with the following combinatorial characterization.

Theorem 2.4. [38, Theorem 3.3] *The map π restricted to $X(w)$ a $X(u)$-fiber bundle if and only if u is maximal length in $[e, w] \cap W_P$.*

We say a parabolic decomposition $w = vu$ is a *Billey-Postnikov (or BP)* decomposition if w satisfies either condition in Theorem 2.4. BP decompositions have become an important combinatorial tool used to study not only Schubert varieties, but Coxeter groups, hyperplane arrangements, and permutation pattern avoidance. Oh and I have recently written a survey article on BP decompositions and their applications in [30].

For rationally smooth Schubert varieties, Slofstra and I prove the following theorem.

Theorem 2.5. [38, Theorem 3.6],[40, Theorem 1.1] *Let $X(w)$ be a Schubert variety of finite type or of affine type A. If $X(w)$ is (rationally) smooth, then w has a BP decomposition with respect to some maximal parabolic subgroup $W_P \subset W$.*

Moreover, if $X(w)$ is smooth, then the morphism $\pi : X(w) \to X^P(v)$ is smooth.

One immediate consequence of Theorem 2.5, is that a smooth Schubert variety in G/B is an iterated fiber bundle of smooth Schubert subvarieties of generalized Grassmannian flag manifolds (G/P where P is maximal). This fact was previously known only in type A [45, 47]. In [38, Theorem 3.8], we give a complete geometric description of smooth Schubert varieties in G/B by classifying all smooth Schubert varieties in generalized Grassmannians. Another consequence is that we prove the Billey-Crites conjecture in [11] which states that smooth Schubert varieties of affine type A correspond to affine permutations avoiding patterns 3412 and 4231.

Our interest in fiber bundle structures of Schubert varieties has its origins in earlier work where Slofstra and I study the combinatorics of Bruhat intervals $[e, w]$ where w is an element of some Coxeter group W. The property that a Schubert variety is smooth can be replaced with the combinatorial notion that the Bruhat interval $[e, w]$ is rank symmetric with respect to length. In other words, the Poincaré polynomial

$$P_w(t) := \sum_{x \in [e,w]} t^{\ell(x)}$$

is a palindromic polynomial. In [37], Slofstra and I show that much of the theory on BP decompositions holds true for several families of Coxeter groups. For example, we prove the following analogue of Theorem 2.5:

Theorem 2.6. [37, Theorem 3.1] *Suppose W has no commuting Coxeter relations. If $P_w(t)$ is palindromic, then w has a BP-decomposition with respect to some proper maximal parabolic subgroup of W.*

We remark that Theorem 2.6 holds for right angled Coxeter groups as well. Theorem 2.6 allows us to construct a combinatorial "fiber bundle" structure on any Coxeter group element with a palindromic Poincaré polynomial. One consequence is that the number of elements for which $P_w(t)$ is palindromic is finite for many infinitely large Coxeter groups [37, Corollary 3.5]. For uniform Coxeter groups, we calculate the generating function for the number of such elements in [37, Proposition 3.8].

2.3. **Enumerating smooth Schubert varieties.** In [39], Slofstra and I develop a model we call *staircase diagrams* over a Dynkin graph which combinatorially encodes the fiber bundle structures of a Schubert variety arising from Theorem 2.5. Our main application is that we calculate the generating function for the number of smooth and rationally smooth Schubert varieties of any classical finite type. This generating function was previously only known in type A and was computed by Haiman [12, 20]. Specifically, define generating series

$$A(t) := \sum_{n=0}^{\infty} a_n\, t^n,\ B(t) := \sum_{n=0}^{\infty} b_n\, t^n,\ C(t) := \sum_{n=0}^{\infty} c_n\, t^n,\ D(t) := \sum_{n=3}^{\infty} d_n\, t^n,\ BC(t) := \sum_{n=0}^{\infty} bc_n\, t^n,$$

where the coefficients a_n, b_n, c_n, d_n denote the number of smooth Schubert varieties of types A_n, B_n, C_n and D_n respectively, and bc_n denotes the number of rationally smooth Schubert varieties of either type B_n or C_n.

Theorem 2.7. [39, Theorem 1.1] *Let $W(t) := \sum_n w_n\, t^n$ denote one of the above generating series, where $W = A$, B, C, D, or BC. Then*

$$W(t) = \frac{P_W(t) + Q_W(t)\sqrt{1-4t}}{(1-t)^2(1-6t+8t^2-4t^3)}$$

where $P_W(t)$ and $Q_W(t)$ are polynomials given in Table 1.

Type	$P_W(t)$	$Q_W(t)$
A	$(1-4t)(1-t)^3$	$t(1-t)^2$
B	$(1-5t+5t^2)(1-t)^3$	$(2t-t^2)(1-t)^3$
C	$1-7t+15t^2-11t^3-2t^4+5t^5$	$t-t^2-t^3+3t^4-t^5$
D	$(-4t+19t^2+8t^3-30t^4+16t^5)(1-t)^2$	$(4t-15t^2+11t^3-2t^5)(1-t)$
BC	$1-8t+23t^2-29t^3+14t^4$	$2t-6t^2+7t^3-2t^4$

TABLE 1. Polynomials in Theorem 2.7.

In [40, Theorem 1.2], Slofstra and I prove an analogous result for the generating function of smooth Schubert varieties of affine type A. One surprising consequence of these enumerations is that the asymptotic growth rate for the number of Schubert varieties is the same for each of the classical Lie types.

2.4. **Fiber bundle structures and pattern avoidance.** For Schubert varieties of finite type A, permutation pattern avoidance has been used to characterize many geometric properties. Most notably, Lakishmbai and Sandhya prove that a Schubert variety is smooth if and only if its corresponding permutation avoids the patterns 3412 and 4231 [26]. Since then, pattern avoidance has been used to characterize other properties such as being defined by inclusions [18], factorial [13] and a local complete intersection [46]. These results have been surveyed by Abe and Billey in [1]. In [2], Alland and I ask the following question:

Question 2.8. *Does the Coxeter theoretic condition for a fiber bundle in Theorem 2.4 have a pattern avoidance characterization in type A?*

We answer this question by developing a new notion of pattern avoidance called *split pattern avoidance*. Let $\mathrm{Fl}(n)$ denote the complete flag variety on $\mathbb{C}^n$ and $\mathrm{Gr}(r,n)$ denote the Grassmannian of r-dimensional subspaces of $\mathbb{C}^n$. There is a natural projection map

$$\pi_r : \mathrm{Fl}(n) \to \mathrm{Gr}(r,n)$$

given by projection onto the r-th factor $\pi(V_\bullet) := V_r$. In this case, Schubert varieties $X(w)$ in $\mathrm{Fl}(n)$ are indexed by permutations.

Theorem 2.9. [2, Theorem 1.1] *Let $w \in W$ be a permutation. The map π_r restricted to $X(w)$ is a fiber bundle if and only if w avoids the split patters $23|1$ and $3|12$ with respect to position r.*

One consequence is that we give a usual pattern avoidance characterization of Schubert varieties with complete parabolic bundle structures.

Theorem 2.10. [2, Theorem 1.3] *Let $w \in W$ be a permutation. Then $X(w)$ has a complete parabolic bundle structure if and only if w avoids the patterns 3412, 52341, 635241.*

In [19], Grigsby and I solve the corresponding enumerative problem on the split patterns $23|1$ and $3|12$.

Theorem 2.11. [19, Theorem 1.2] *Let $k(r,n)$ denote the number of permutations of type A_{n-1} that avoid $23|1$ and $3|12$ with respect to position r. Then*

$$k(r,n) = r!(n-r)! + \sum_{i=r}^{r} \sum_{j=1}^{n-r} \binom{n-i-j}{r-i} \cdot (r)_{i-1} \cdot (n-r)_{j-1}$$

where $(m)_i := m(m-1)\cdots(m-i+1)$ denotes falling factorial.

Grigsby and I also calculate the bivariate generating function of the values $k(r,n)$ and show that there is a nice connection with Bessel functions which are solutions to the Bessel partial differential equation.

2.5. **The Nash blowup of a Schubert variety.** The Nash blow-up of a complex algebraic variety is the parameter space of tangent spaces over its smooth locus together with the limits of tangents spaces over its singular locus. One motivation for studying the Nash blow-up is that its tautological bundle serves as an analogue of the tangent bundle for singular varieties. The existence of such a blow-up has led to the development of a characteristic class theory for singular varieties [28]. For Schubert varieties, these classes have been extensively studied in [3, 4, 5, 22]. While the Nash blow-up is an extremely important object in class theory, its geometry and combinatorics is poorly understood. In [42], Slofstra, Woo and I calculate the Nash blow-up of cominuscule Schubert varieties and show that the torus-fixed points of the Nash blow-up correspond to Peterson translates of the inversion set. This work is inspired by earlier work by Carrell and Kuttler in [15] where they define Peterson translation on T-stable varieties and use it to determine when a T-fixed point in the Schubert variety is smooth.

Theorem 2.1. [42, Theorem 2.1] *Let Δ, Δ_P denote the set of simple roots for G and P respectively and let $X(w)$ be a cominuscule Schubert variety in G/P. Further assume that w in minimal length in the coset wW_P.*

Then the Nash blow-up $X(w)$ is a Schubert variety. In particular, it is algebraically isomorphic to $\overline{BwQ}/Q$ for the standard parabolic subgroup $Q \subseteq P$, where Q is generated by the set of simple roots

$$\Delta_w := \{\beta \in \Delta_P \mid w(\beta) \in \Delta\}.$$

Theorem 2.1 has many consequences. First, it immediately implies that the Nash blow-up of $X(w)$ is a normal variety. Second, we use this result to give a new characterization of the smooth locus of $X(w)$. For Grassmannian Schubert varieties (which are all cominuscule), we determine when the Nash blow-up is a resolution of singularities. We also show that the Nash blow-up is a fiber product of left-peak and right-peak Zelevinsky resolutions.

3. Schubert calculus

The goal of Schubert calculus is understand the product structure of various cohomology theories of flag varieties and their generalizations with respect to a basis of Schubert classes. Questions can either be geometric or combinatorial in nature. In this section, I will discuss my research projects on Schubert calculus.

3.1. Grassmannian Schubert calculus and applications.

This section is about two projects involving Schubert calculus of the Grassmannian $\mathrm{Gr}(r,n)$ of r-dimensional subspaces in $\mathbb{C}^n$. The cohomology ring $H^*(\mathrm{Gr}(r,n))$ has an additive basis of Schubert classes $\{\sigma_\lambda\}_{\lambda\in\Lambda}$, where Λ is the set of partitions whose Young diagrams are contained in an $r\times(n-r)$ rectangle. For any three partitions $\lambda,\mu,\nu\in\Lambda$ we can define the Littlewood-Richardson coefficients $c^\nu_{\lambda,\mu}$ by the product structure constants

$$\sigma_\lambda\cdot\sigma_\mu=\sum_{\nu\in\Lambda}c^\nu_{\lambda,\mu}\,\sigma_\nu. \tag{1}$$

The Littlewood-Richardson coefficients arise in several fields of mathematics including the representation theory of the general linear group, the combinatorics of symmetric functions, and quiver representations.

One remarkable application of Littlewood-Richardson coefficients is to the eigenvalue problem on sums of hermitian matrices. The following theorem is proved by the combined works of Klyachko [23] and Knutson and Tao [24].

Theorem 3.1. ([23, 24]) *The coefficient $c^\nu_{\lambda,\mu}>0$ if and only if there exist $r\times r$ hermitian matrices A,B,C with eigenvalues given by the partitions λ,μ,ν and*

$$A+B=C.$$

In joint work with Anderson and Yong [6], we are able to extend this result to the setting of torus-equivariant cohomology of the Grassmannian $H^*_T(\mathrm{Gr}(r,n))$. Define the structure constants $C^\nu_{\lambda,\mu}$ by the product of equivariant Schubert classes

$$\Sigma_\lambda\cdot\Sigma_\mu=\sum_{\nu\in\Lambda}C^\nu_{\lambda,\mu}\,\Sigma_\nu.$$

We have the following theorem (omitting some technical constraints).

Theorem 3.2. [6, Theorem 1.3] *The coefficient $C^\nu_{\lambda,\mu}>0$ if and only if there exist $r\times r$ hermitian matrices A,B,C with eigenvalues given by the partitions λ,μ,ν and*

$$A+B\geq C.$$

Here a matrix $A\geq B$ if $A-B$ is positive semi-definite. Theorem 3.2 is proved by showing that Horn's inequalities, which determine when $c^\nu_{\lambda,\mu}>0$, also determine when $C^\nu_{\lambda,\mu}>0$ in the equivariant setting. As a corollary, we get an equivariant generalization of the celebrated saturation theorem.

Theorem 3.3. [6, Thoerem 1.1] *$C^\nu_{\lambda,\mu}>0$ if and only if $C^{N\nu}_{N\lambda,N\mu}>0$ for any $N>0$.*

Another application of Theorem 3.1 is to frame theory, an important topic in functional analysis. Let $P_1,\ldots,P_k$ be a sequence of $N\times N$ orthogonal projection matrices and let

$\mathbf{L} := (L_1, \dots, L_k)$ denote the corresponding rank sequence (i.e. $\mathrm{rank}(P_i) = L_i$). We say that $P_1, \dots, P_k$ is a *tight fusion frame* if there exists a real number α such that

$$\sum_{i=1}^{k} P_i = \alpha \mathbf{I}_N$$

where $\mathbf{I}_N$ denotes the identity matrix. Applications of fusion frames include sensor networks, coding theory, compressed sensing, and filter banks. In [14], together with Bownik and Luoto, we address the problem of classifying all sequences $\mathbf{L}$ that are rank sequences of some tight fusion frame. Since orthogonal projection matrices are hermitian, we use Theorem 3.1 to prove the following classification.

Theorem 3.4. [14, Theorem 4.2] $\mathbf{L} = (L_1, \dots, L_k)$ *is a tight fusion frame sequence if and only if*

$$\prod_{i=1}^{k} \sigma_{(N^{L_i})} \neq 0$$

in $H^*(\mathrm{Gr}(N, M+N))$ *where* $M := \sum_{i=1}^{k} L_i$ *and the partition* $(N^{L_i}) := \underbrace{(N, \dots, N)}_{L_i}$.

This connection between frame theory and Schubert calculus yields many interesting results in both fields of mathematics. For example, using Schubert combinatorics, we produce new bounding estimates on tight fusion frames previously unknown in frame theory. Conversely, inspired by dualities found in frame theory, we construct new combinatorial identities for Littlewood-Richardson coefficients.

3.2. **Schubert calculus for Kac-Moody groups.** In joint work with Berenstein from [9], we study the Schubert calculus of the flag variety G/B corresponding to a Kac-Moody group G. The structure of G is encoded by a *generalized Cartan matrix* (GCM), defined to be a square matrix $A = (a_{i,j})$ where $a_{i,i} = 2$ and $a_{i,j} \in \mathbb{Z}_{<0}$ if $i \neq j$. Thus for each GCM, we can associate and study the cohomology ring $H^*(G/B)$.

Like the cohomology of the Grassmannian, $H^*(G/B)$ has an additive basis of Schubert classes indexed by W, the Weyl group of G. We define the structure constants $c_{u,v}^{w}$ by the product

$$\sigma_u \cdot \sigma_v = \sum_{w \in W} c_{u,v}^{w} \, \sigma_w.$$

In [9, Theorem 2.4], we give a formula for computing $c_{u,v}^{w}$ in terms of the GCM A. This formula is based on the work of Kostant and Kumar in [25] where they study nil-Hecke rings corresponding to Kac-Moody groups. While other formulas for Schubert structure constants exist (see [16]), it has been a long-standing open problem to find a formula that is "combinatorially positive". Although it is well known from the geometry of G/B that the Schubert structure constants are non-negative integers, there are no known combinatorial proofs of this positivity (except in a few very special cases). Our formula satisfies the following property.

Theorem 3.5. [9, Theorem 2.16] *If the GCM* $A = (a_{i,j})$ *of* G *satisfies*

$$a_{i,j} a_{j,i} \geq 4 \tag{2}$$

for all i, j, *then the formula for* $c_{u,v}^{w}$ *given in* [9, Theorem 2.4] *is combinatorially positive.*

In other words, the formula we construct is completely algebraic and the proof of positivity does not rely on the geometry of G/B. The condition (2) is precisely the condition that the Weyl group W has no braid relations or commuting relations as a Coxeter group. Theorem 3.5 above and [9, Theorem 2.4] have both been extended to include Schubert structure constants for the torus-equivariant cohomology $H_T^*(G/B)$ in [9]. Recently, Zainoulline and I have written a survey article on nil-Hecke rings and their applications to Schubert calculus [44]. This article is written at the generality of real reflection groups acting on non-crystallographic root-datum which much of the algebraic theory still holds.

3.3. **Recursive formulas for structure constants.** Let $P \subseteq Q$ be a pair of parabolic subgroups of a complex Lie group G and consider the induced sequence of partial flag varieties

$$Q/P \hookrightarrow G/P \twoheadrightarrow G/Q.$$

When comparing the three flag varieties above, the variety G/P typically has the most complicated cohomology structure. In [34, 35], I develop a recursive formula to compute Schubert structure coefficients of $H^*(G/P)$ in terms of the simpler cohomology rings $H^*(Q/P)$ and $H^*(G/Q)$ under certain constraints.

Theorem 3.6. [35, Theorem 1.1] *Let $(w_1, w_2, w_3) \in (W^P)^3$ with parabolic decompositions $w_i = v_i u_i$ with respect to Q. If the triples (w_1, w_2, w_3) and (v_1, v_2, v_3) satisfy a certain numerical constraint, then*

$$c_{w_1,w_2}^{w_3} = c_{v_1,v_2}^{v_3} \cdot c_{u_1,u_2}^{u_3}.$$

One important class of coefficients satisfying these constraints of [35, Theorem 1.1] are coefficients $c_{u,v}^w$ corresponding to *Levi-movable* triples (u, v, w) defined by Belkale and Kumar [8]. In [32], Ressayre shows that the set of Levi-movable triples, with $c_{u,v}^w = 1$, indexes the interior faces of the eigencone corresponding to the group G. By applying the recursive formula [35, Theorem 1.1] to Ressayre's work, I am able to determine the inclusion relations of the faces of the eigencone.

In [33], Ressayre and I generalize the notion of Levi-movability to the setting of "branching Schubert calculus". Branching Schubert calculus refers to the problem of computing the comorphism on cohomology rings induced from an equivariant embedding of one flag variety into another. If we consider the diagonal embedding of a flag variety into two copies of itself, then the comorphism on cohomology is simply the cup product. Hence, branching Schubert calculus is a generalization of usual Schubert calculus. We use the generalized definition of Levi-movable to give a more elegant solution to the branching eigenvalue problem.

The main idea behind the proof of the recursive formula [35, Theorem 1.1] and its various applications to Levi-movability is to use the fact that Schubert structure coefficients count the number of points in the intersection of corresponding sets of Schubert varieties in general position. Since this intersection is transverse, we can apply tangent space analysis.

3.4. **Noncommutative Littlewood-Richardson coefficients.** For any composition α, let s_α denote the noncommutative Schur function as defined in [10]. The *noncommutative*

Littlewood-Richardson coefficients $C^{\gamma}_{\alpha,\beta}$ are defined as the structure coefficients of the product

$$\mathbf{s}_\alpha \cdot \mathbf{s}_\beta = \sum_\gamma C^{\gamma}_{\alpha,\beta}\, \mathbf{s}_\gamma$$

in the algebra of noncommutative symmetric functions. Bessenrodt, Luoto and vanWilligenburg prove in [10] that the coefficients $C^{\gamma}_{\alpha,\beta}$ are nonnegative integers and are refinements of classical LR coefficients. More precisely, [10, Corollary 3.7] states for any compositions α, β with underlying partitions shapes $\lambda = \tilde{\alpha}$ and $\mu = \tilde{\beta}$, we have

$$c^{\nu}_{\lambda,\mu} = \sum_{\tilde{\gamma}=\lambda} C^{\gamma}_{\alpha,\beta}.$$

Note that these classical coefficients are the same those that the determine the Schubert calculus of the Grassmannian given in Equation (1). Combinatorially, it is well known that the coefficient $c^{\nu}_{\lambda,\mu}$ counts the number of LR skew-tableaux of shape ν/λ with content μ. Let $\mathrm{LRT}(\lambda,\mu,\nu)$ denote the set of such tableaux. In [43], Tewari and I prove the following decomposition theorem.

Theorem 3.1. *Given compositions α and β such that $\lambda = \tilde{\alpha}$ and $\mu = \tilde{\beta}$, there exists a natural decomposition*

$$\mathrm{LRT}(\lambda,\mu,\nu) = \bigsqcup_{\tilde{\gamma}=\lambda} X^{\gamma}_{\alpha,\beta}$$

such that $|X^{\gamma}_{\alpha,\beta}| = C^{\gamma}_{\alpha,\beta}$.

Theorem 3.1 is a tableaux analogue of Equation (3.4). To determine if a tableaux belongs to $X^{\gamma}_{\alpha,\beta}$, we apply a sequence of crystal reflection operators corresponding to a reduced word of a permutation σ such that $\sigma \cdot \lambda = \alpha$. We then apply Mason's map indexed by β which reveals a composition shape γ [29]. The set $X^{\gamma}_{\alpha,\beta}$ is defined as the set of LR skew tableaux whose image under this process yields the shape γ.

4. OTHER RESEARCH PROJECTS

The following are some additional research projects I have recently worked on.

4.1. **Cohomology of Springer fibers.**

In [31], Precup and I study the geometry and topology of Springer fibers. In particular, we construct a T-equivariant analogue of the Garsia-Procesi (GP) basis for the cohomology of Springer fibers given in [17]. We call this new basis the *equivariant Springer monomials*. The main application of this basis is that it provides a combinatorial framework to make calculations in cohomology of the Springer fiber. For example, we compute the pull back of a Schubert class in the cohomology. As a consequence, we show that there always exists a successful game of Betti poset pinball for type A Springer fibers. Existence of such games was questioned by Harada and Tymoczko in [21].

4.2. **Intervals in Young's lattice.**

In [7], Azam and I study lower order ideals in Young's lattice on partitions. Consider the polynomial

$$P_\lambda(y) := \sum_{\mu \leq \lambda} y^{|\mu|}$$

where $|\mu|$ denotes the size of μ. Topologically, $P_\lambda(y)$ is the Poincaré polynomial of the Grassmannian Schubert variety $X(\lambda)$. Define the generating function

$$Q_k(x_1, \ldots, x_k, y) := \sum_{\lambda \in \Lambda(k)} P_\lambda(y) \cdot x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$$

where $\Lambda(k)$ denotes the set of partitions with exactly k parts. Azam and I prove the following theorem:

Theorem 4.1. *The function Q_k is a rational function in the variables $x_1, \ldots, x_k, y$.*

One application of Theorem 4.1 is that we can compute the growth rate of the coefficients at various deformations of Q_k. For example, we show that the generating function $Q_k(x, \ldots, x, 1)$ only has singularities of modulus one and hence the corresponding coefficients have polynomial growth. These coefficients correspond to the "average size" of a lower order ideal in Young's lattice.

4.3. **Demazure products and hopping.** Given a Coxeter system (W, S), the Demazure product $\star$ is defined as the Coxeter moniod structure derived by replacing the nil-relation $s \cdot s = e$ by the relation $s \star s = s$ for $s \in S$. This product arises naturally in study of Hecke algebras and their corresponding Lie groups. For example, if W is the Weyl group of a reductive group G, then the Borel orbit relations are given by

$$\overline{BwBuB} = \overline{B(w \star u)B}.$$

In [27], together with Li, Oh, Yan and You, we study the Demazure product on permutations. Our main result is an efficient algorithm for calculating the Demazure product using only the one-line notation of permutations. To describe this algorithm, we define a new operator called a *hopping operator* on permutations. We also extend our hopping algorithm to the group of signed permutations which are Coxeter groups of type B/C.

References

[1] Hiraku Abe and Sara Billey. Consequences of the Lakshmibai-Sandhya theorem: the ubiquity of permutation patterns in Schubert calculus and related geometry. In *Schubert calculus—Osaka 2012*, volume 71 of *Adv. Stud. Pure Math.*, pages 1–52. Math. Soc. Japan, [Tokyo], 2016.

[2] Timothy Alland and Edward Richmond. Pattern avoidance and fiber bundle structures on Schubert varieties. *J. Combin. Theory Ser. A*, 154:533–550, 2018.

[3] Paolo Aluffi and Leonardo C. Mihalcea. Chern-Schwartz-MacPherson classes for Schubert cells in flag manifolds. *Compos. Math.*, 152(12):2603–2625, 2016.

[4] Paolo Aluffi, Leonardo C. Mihalcea, Joerg Schuermann, and Changjian Su. Shadows of characteristic cycles, verma modules, and positivity of chern-schwartz-macpherson classes of schubert cells. *preprint, arXiv:1709.08697*.

[5] Paolo Aluffi and Leonardo Constantin Mihalcea. Chern classes of Schubert cells and varieties. *J. Algebraic Geom.*, 18(1):63–100, 2009.

[6] David Anderson, Edward Richmond, and Alexander Yong. Eigenvalues of Hermitian matrices and equivariant cohomology of Grassmannians. *Compos. Math.*, 149(9):1569–1582, 2013.

[7] Faqruddin Ali Azam and Edward Richmond. On the generating function for intervals in Young's lattice. *Electron. J. Combin.*, 30(2):Paper No. 2.16, 19, 2023.

[8] Prakash Belkale and Shrawan Kumar. Eigenvalue problem and a new product in cohomology of flag varieties. *Invent. Math.*, 166(1):185–228, 2006.

[9] Arkady Berenstein and Edward Richmond. Littlewood-Richardson coefficients for reflection groups. *Adv. Math.*, 284:54–111, 2015.

[10] C. Bessenrodt, K. Luoto, and S. van Willigenburg. Skew quasisymmetric Schur functions and noncommutative Schur functions. *Adv. Math.*, 226(5):4492–4532, 2011.

[11] Sara Billey and Andrew Crites. Pattern characterization of rationally smooth affine Schubert varieties of type A. *J. Algebra*, 361:107–133, 2012.
[12] Miklós Bóna. The permutation classes equinumerous to the smooth class. *Electron. J. Combin.*, 5:Research Paper 31, 12 pp. (electronic), 1998.
[13] Mireille Bousquet-Mélou and Steve Butler. Forest-like permutations. *Ann. Comb.*, 11(3-4):335–354, 2007.
[14] Marcin Bownik, Kurt Luoto, and Edward Richmond. A combinatorial characterization of tight fusion frames. *Pacific J. Math.*, 275(2):257–294, 2015.
[15] James B. Carrell and Jochen Kuttler. Smooth points of T-stable varieties in G/B and the Peterson map. *Inventiones Mathematicae*, 151(2):353–379, 2003.
[16] Haibao Duan. Multiplicative rule of Schubert classes. *Invent. Math.*, 159(2):407–436, 2005.
[17] A. M. Garsia and C. Procesi. On certain graded S_n-modules and the q-Kostka polynomials. *Adv. Math.*, 94(1):82–138, 1992.
[18] V. Gasharov and V. Reiner. Cohomology of smooth Schubert varieties in partial flag manifolds. *J. London Math. Soc. (2)*, 66(3):550–562, 2002.
[19] Travis Grigsby and Edward Richmond. Enumerating permutations avoiding split patterns $3|12$ and $23|1$. *preprint, arXiv:2402.17654*, 2024.
[20] Mark Haiman. Enumeration of smooth schubert varieties. *preprint*, 1992.
[21] Megumi Harada and Julianna Tymoczko. Poset pinball, GKM-compatible subspaces, and Hessenberg varieties. *J. Math. Soc. Japan*, 69(3):945–994, 2017.
[22] Benjamin F. Jones. Singular Chern classes of Schubert varieties via small resolution. *Int. Math. Res. Not. IMRN*, (8):1371–1416, 2010.
[23] Alexander A. Klyachko. Stable bundles, representation theory and Hermitian operators. *Selecta Math. (N.S.)*, 4(3):419–445, 1998.
[24] Allen Knutson and Terence Tao. The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.*, 12(4):1055–1090, 1999.
[25] Bertram Kostant and Shrawan Kumar. The nil Hecke ring and cohomology of G/P for a Kac-Moody group G. *Adv. in Math.*, 62(3):187–237, 1986.
[26] V. Lakshmibai and B. Sandhya. Criterion for smoothness of Schubert varieties in $\mathrm{Sl}(n)/B$. *Proc. Indian Acad. Sci. Math. Sci.*, 100(1):45–52, 1990.
[27] Tina Li, Suho Oh, Edward Richmond, Grace Yan, and Kimberly You. Demazure product of permutations and hopping. *Electron. J. Combin.*, 31(1):Paper No. 1.48, 15, 2024.
[28] R. D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.
[29] S. Mason. An explicit construction of type A Demazure atoms. *J. Algebraic Combin.*, 29(3):295–313, 2009.
[30] Suho Oh and Edward Richmond. Coxeter groups and billey-postnikov decompositions. *preprint, arXiv:2409.03096*, 2024.
[31] Martha Precup and Edward Richmond. An equivariant basis for the cohomology of Springer fibers. *Trans. Amer. Math. Soc. Ser. B*, 8:481–509, 2021.
[32] Nicolas Ressayre. Geometric invariant theory and the generalized eigenvalue problem. *Invent. Math.*, 180(2):389–441, 2010.
[33] Nicolas Ressayre and Edward Richmond. Branching Schubert calculus and the Belkale-Kumar product on cohomology. *Proc. Amer. Math. Soc.*, 139(3):835–848, 2011.
[34] Edward Richmond. A partial Horn recursion in the cohomology of flag varieties. *J. Algebraic Combin.*, 30(1):1–17, 2009.
[35] Edward Richmond. A multiplicative formula for structure constants in the cohomology of flag varieties. *Michigan Math. J.*, 61(1):3–17, 2012.
[36] Edward Richmond, Mihail Ţarigradschi, and Weihong Xu. The isomorphism problem for cominuscule Schubert varieties. *Selecta Math. (N.S.)*, 30(3):Paper No. 38, 17, 2024.
[37] Edward Richmond and William Slofstra. Rationally smooth elements of Coxeter groups and triangle group avoidance. *J. Algebraic Combin.*, 39(3):659–681, 2014.
[38] Edward Richmond and William Slofstra. Billey-Postnikov decompositions and the fibre bundle structure of Schubert varieties. *Math. Ann.*, 366(1-2):31–55, 2016.
[39] Edward Richmond and William Slofstra. Staircase diagrams and enumeration of smooth Schubert varieties. *J. Combin. Theory Ser. A*, 150:328–376, 2017.
[40] Edward Richmond and William Slofstra. Smooth Schubert varieties in the affine flag variety of type $\tilde{A}$. *European J. Combin.*, 71:125–138, 2018.

[41] Edward Richmond and William Slofstra. The isomorphism problem for schubert varieties. *preprint, arXiv:2103.08114*, 2021.

[42] Edward Richmond, William Slofstra, and Alexander Woo. The Nash blow-up of a cominuscule Schubert variety. *J. Algebra*, 559:580–600, 2020.

[43] Edward Richmond and Vasu Tewari. Noncommutative LR coefficients and crystal reflection operators. *Algebr. Comb.*, 4(1):145–162, 2021.

[44] Edward Richmond and Kirill Zainoulline. Nil-hecke rings and the schubert calculus. *preprint, arXiv:2310.01167*, 2023.

[45] Kevin M. Ryan. On Schubert varieties in the flag manifold of $\mathrm{Sl}(n, \mathbb{C})$. *Mathematische Annalen*, 276(2):205–224, 1987.

[46] Henning Ulfarsson and Alexander Woo. Which Schubert varieties are local complete intersections? *Proc. Lond. Math. Soc. (3)*, 107(5):1004–1052, 2013.

[47] James S Wolper. A combinatorial approach to the singularities of Schubert varieties. *Advances in Mathematics*, 76(2):184–193, 1989.